BAYESIAN ESTIMATION

Until now discussion about estimation has assumed a frequentist approach, namely:

- o The parameter of the population distribution is unknown but fixed (not random);

- o The inference procedures are based not only on the observed sample but also on the population of samples that could have been observed.

- o The Bayesian approach assumes that our lack of knowledge about the parameters value should be translated using probability distributions (consequently **unknown parameters are treated as random variables**) and that **only the observed data** (and not the population of samples) **is relevant** to make statistical inference.

Bayesian Inference

$$
\left.\begin{array}{l} \text{Model } f_{X|\Theta}(x\,|\,\theta) \\ \text{Sample } (x_1, x_2, \cdots, x_n) \end{array}\right\} \rightarrow \left.\begin{array}{l} \text{Model distribution } f_{\mathbf{X}|\Theta}(\mathbf{x}\,|\,\theta) \\ \text{Prior distribution } \pi(\theta) \end{array}\right\} \begin{array}{l} \text{Bayes} \\ \rightarrow \\ \text{Theorem} \end{array} \text{Posterior distribution } \pi_{\Theta|\mathbf{x}}(\theta\,|\,\mathbf{x})
$$

Prior distribution

o **Definition 15.1** – The **prior distribution** is a probability distribution over the space of possible parameter values. It is denoted $\pi(\theta)$ and represents our opinion concerning the relative chances that various values of θ are the true value of the parameter.

o **Comments:**

  o The existence of a **prior** for θ (scalar or vector) is the core of **Bayesian inference**. From a theoretical point of view it raises important questions about the concept of probability.

  o From a practical point of view, the determination of the prior is a **major problem** of Bayesian methods. In many situations we have some insights about possible parameter values but the main difficulty is translating this knowledge into a probability distribution.

  o Due to the difficulty of finding a prior, we often use an **improper prior** distribution (vague prior) or we take advantage of **conjugate priors**.

o **Definition 15.2** – An **improper prior distribution** is one for which the probabilities (or probability density function) are nonnegative but their sum (integral) is infinite.

o **Comments:**

  o The improper prior is one possible solution when we have minimal knowledge about the parameter behavior.

  o Universal agreement on the best way to construct a vague (or non-informative) prior does not exist.

  o However the use of the improper prior $\pi(\theta) = 1/\theta$, $\theta > 0$ as a **vague prior** for a scale parameter is quite consensual.

o **Definition 15.17** – A prior distribution is said to be a conjugate prior distribution for a given model if the resulting posterior distribution is from the same family as the prior (but perhaps with different parameters).

Model distribution

- **Theoretical model for the population:** for instance Bernoulli, normal, ….

- **Instead of considering a random sample** $\mathbf{X} = \left(X_1, X_2, \cdots, X_n \right)$ – usually the sampling process generates i.i.d. observations – we only look at the observed sample $\mathbf{x} = \left(x_1, x_2, \cdots, x_n \right)$

- **Definition 15.3** – The **model distribution** is the probability distribution for the data as collected given a particular value for the parameter. Note that this matches definition 13.4 for the likelihood function. However, consistent with Bayesian notion, the model pdf is denoted $f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)$, where vector notation for $\mathbf{x}$ is used to remind us that all the data appear here.

- **Comments:**

  o If the observations are i.i.d., then $L(\theta \mid \mathbf{x}) = f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta) = \prod_{i=1}^{n} f_{\mathbf{X}|\Theta}(x_1 \mid \theta)$.

  o Only the likelihood of the **observed sample** is relevant

Bayes Theorem: How to obtain the posterior distribution?

o **Definition 15.6** – The **posterior distribution** is the conditional probability distribution of the parameters, given the observed data. It is denoted $\pi_{\Theta|\mathbf{X}}(\theta\,|\,\mathbf{x})$.

o **Theorem 15.8** – (Part a) The posterior distribution can be computed as $\pi_{\Theta|\mathbf{X}}(\theta\,|\,\mathbf{x}) = \dfrac{f_{\mathbf{X}|\Theta}(\mathbf{x}\,|\,\theta)\times\pi(\theta)}{\int f_{\mathbf{X}|\Theta}(\mathbf{x}\,|\,\theta)\times\pi(\theta)\,d\theta}$

o **Comments:**

  o This is the central purpose of Bayesian analysis: The posterior distribution represents our beliefs about θ once the sample has been observed (and for a given prior).

  o In most cases we determine the posterior up to a normalizing constant, i.e. $\pi_{\Theta|\mathbf{X}}(\theta\,|\,\mathbf{x}) \propto f_{\mathbf{X}|\Theta}(\mathbf{x}\,|\,\theta)\times\pi(\theta)$ or $\pi_{\Theta|\mathbf{X}}(\theta\,|\,\mathbf{x}) \propto L(\theta\,|\,\mathbf{x})\times\pi(\theta)$ as $f_{\mathbf{X}|\Theta}(\mathbf{x}\,|\,\theta) = L(\theta\,|\,\mathbf{x})$.

  o This constant can be determined afterwards using the condition $\int \pi_{\Theta|\mathbf{X}}(\theta\,|\,\mathbf{x})\,d\theta = 1$ but, when the posterior belongs to a known family of distributions, we can avoid any computation: we identify the core of the family and then we get the constant (using for instance Appendix A or B of the book).

  o Remember Bayes's formula: Partition $\{A_1, A_2, \cdots\}$, event B, then $P(A_i\,|\,B) = \dfrac{P(B\,|\,A_i)\times P(A_i)}{\sum_i P(B\,|\,A_i)\times P(A_i)}$

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Example – Given that we observed the sample $(0,1,1,0,1)$ from a population $X \mid \theta \sim Ber(\theta)$ and that the prior is $\pi(\theta) = 1$, $0 < \theta < 1$, obtain the posterior distribution.

$$\pi(\theta) = 1, \ 0 < \theta < 1 \qquad\qquad L(\theta \mid \mathbf{x}) = \prod_{i=1}^{5} \theta^{x_i} (1-\theta)^{1-x_i} = \theta^3 (1-\theta)^2, \ 0 < \theta < 1$$

Then, $\pi_{\Theta \mid \mathbf{X}}(\theta \mid \mathbf{x}) \propto \theta^3 (1-\theta)^2$, $0 < \theta < 1$
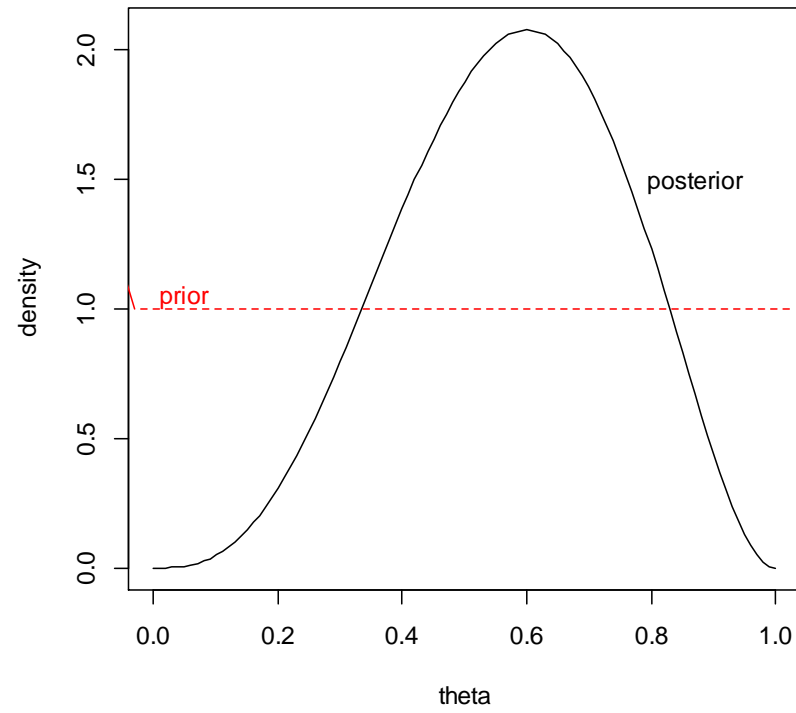
The normalizing constant is given by $\displaystyle\int_0^1 \theta^3 (1-\theta)^2 \, d\theta = \int_0^1 \theta^3 + \theta^5 - 2\theta^4 d\theta = \frac{1}{4} + \frac{1}{6} - \frac{2}{5} = \frac{15 + 10 - 24}{60} = \frac{1}{60}$

and the posterior is $\pi_{\Theta \mid \mathbf{X}}(\theta \mid \mathbf{x}) = 60 \theta^3 (1-\theta)^2$, $0 < \theta < 1$.

If we notice that $\theta^3 (1-\theta)^2$, $0 < \theta < 1$, is the core of a beta distribution with parameters 4 and 3 we can get the normalizing constant as $\dfrac{\Gamma(4+3)}{\Gamma(4) \times \Gamma(3)} = \dfrac{6!}{3! \times 2!} = \dfrac{6 \times 5 \times 4}{2} = 60$, avoiding the computation of the integral.

The predictive distribution

- **Definition 15.7** – The predictive distribution is the conditional distribution of a new observation y given the data $\mathbf{x}$. It is denoted $f_{Y|\mathbf{X}}(y \mid \mathbf{x})$

- **Theorem 15.8** – (Part b) The predictive distribution can be computed as

$$f_{Y|\mathbf{X}}(y \mid \mathbf{x}) = \int f_{Y|\Theta}(y \mid \theta)\, \pi_{\Theta|\mathbf{X}}(\theta \mid \mathbf{x})\, d\theta$$

  Where $f_{Y|\Theta}(y \mid \theta)$ is the pdf of the new observation, given the parameter value.

- **Example** – Using the previous example, compute the predictive distribution before and after the sample was observed.

  After:

$$f_{Y|\mathbf{X}}(y \mid \mathbf{x}) = \int_0^1 \theta^y (1-\theta)^{1-y}\, 60\theta^3 (1-\theta)^2\, d\theta = 60 \int_0^1 \theta^{y+3} (1-\theta)^{3-y}\, d\theta$$

  As $y = 0$ or $y = 1$ we get

$$f_{Y|\mathbf{X}}(1 \mid \mathbf{x}) = 60 \int_0^1 \theta^4 (1-\theta)^2\, d\theta = 60 \int_0^1 \left(\theta^4 + \theta^6 - 2\theta^5\right) d\theta = 60 \left(\frac{1}{5} + \frac{1}{7} - \frac{2}{6}\right) = 12 + \frac{60}{7} - 20 = \frac{60}{7} - 8 = \frac{4}{7}$$
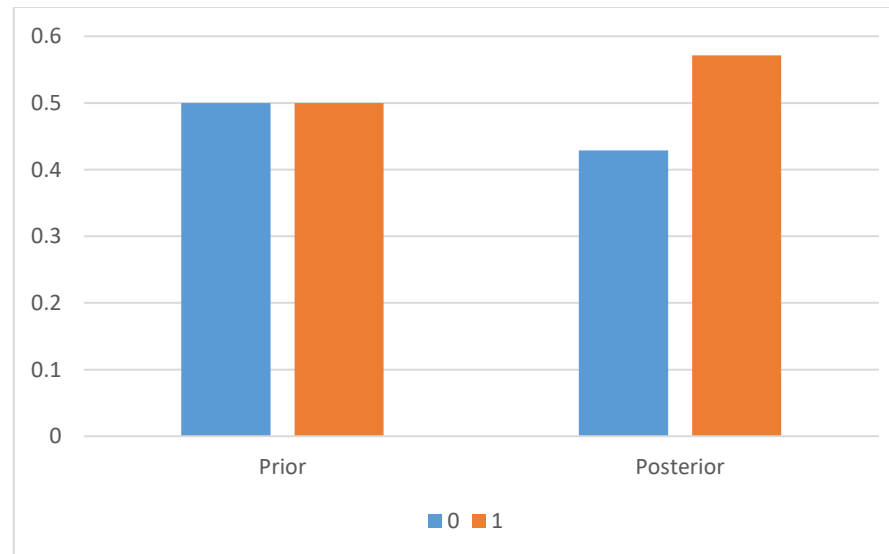
and consequently $f_{Y|\mathbf{X}}(0|\mathbf{x}) = \dfrac{3}{7}$

Before

As the sample was not yet observed we use the prior instead of the posterior

$$f_{\mathbf{X}}(x) = \int_0^1 \theta^y (1-\theta)^{1-y} 1 d\theta = \int_0^1 \theta^y (1-\theta)^{1-y} d\theta$$

Then $f_X(1) = \int_0^1 \theta(1-\theta)^0 \, d\theta = \int_0^1 \theta \, d\theta = 1/2$ and consequently $f_X(0) = 1/2$

Other definitions (less important)

o **Definition 15.4** – The joint distribution has pdf $f_{\mathbf{X},\Theta}(\mathbf{x}\,|\,\theta)\times\pi(\theta)$

o **Definition 15.5** – The marginal distribution of $\mathbf{x}$ has pdf $f_{\mathbf{X}}(\mathbf{x})=\int f_{\mathbf{X},\Theta}(\mathbf{x}\,|\,\theta)\times\pi(\theta)\,d\theta$

Example 15.1 – The following amounts were paid on a hospital liability policy

$$125\ \ 132\ \ 141\ \ 107\ \ 133\ \ 319\ \ 126\ \ 104\ \ 145\ \ 223.$$

The amount of a single payment has the single-parameter Pareto distribution with $\theta=100$ and α unknown. The prior is a gamma distribution with parameters $\alpha=2$ and $\theta=1$. Determine all of the relevant Bayesian quantities.

Prior: $\pi(\alpha)=\dfrac{\alpha^{2-1}\,e^{-\alpha/1}}{\Gamma(2)\,1^2}=\alpha\,e^{-\alpha}$, $\alpha>0$ This means that $\alpha\sim\gamma(2,1)$, $E(\alpha)=\mathrm{var}(\alpha)=2$

Likelihood:

$$L(\alpha \mid \mathbf{x}) = \prod_{i=1}^{n} f(x_i \mid \alpha) = \prod_{i=1}^{n} \frac{\alpha 100^{\alpha}}{x_i^{\alpha+1}} \qquad x_i > 100$$

$$= \alpha^{10} \left(\prod_{i=1}^{10} \frac{100^{\alpha}}{x_i^{\alpha}} \right) \left(\prod_{i=1}^{10} \frac{1}{x_i} \right) = \alpha^{10} \times 0.022346^{\alpha} \times \frac{1}{\prod_{i=1}^{10} x_i} \propto \alpha^{10} \times 0.022346^{\alpha}$$

Posterior:

$$\pi_{A|\mathbf{X}}(\alpha \mid \mathbf{x}) \propto L(\alpha \mid \mathbf{x}) \times \pi(\alpha) \propto \alpha^{10} \times 0.022346^{\alpha} \times \alpha \times e^{-\alpha}$$

$$= \alpha^{11} \times \exp\left(\alpha \ln 0.022346\right) \times e^{-\alpha} = \alpha^{11} e^{-\alpha(1 - \ln 0.022346)}$$

$$= \alpha^{11} e^{-4.80112\alpha} \qquad \alpha > 0$$

We get the *core* of a gamma distribution with parameters 12 and 1/4.80112 and then we know that the normalizing constant is $4.80112^{12} / \Gamma(12) = 3.757995$. As the posterior belongs to the same family of the prior we said that we are using a conjugate prior for this model.

The point here is that the observed samples leads us to change our believes about α from a $\gamma(2,1)$ to a $\gamma(12, 0.20828)$ and now $E(\alpha \mid \mathbf{x}) = 2.49942$ and $\text{var}(\alpha \mid \mathbf{x}) = 0.52059$.

We can draw both densities on the same graph to visualize the differences:

```
> x=seq(0,6,by=0.2)
> plot(x,dgamma(x,shape=1,scale=1),type="l",ylab="density",xlab="alpha")
> y=dgamma(x,shape=12,scale=0.208285) # posterior
> lines(x,y,type="l",lty=2)
> text(3.5,0.45,"posterior"); text(0.8,0.7,"prior")
```
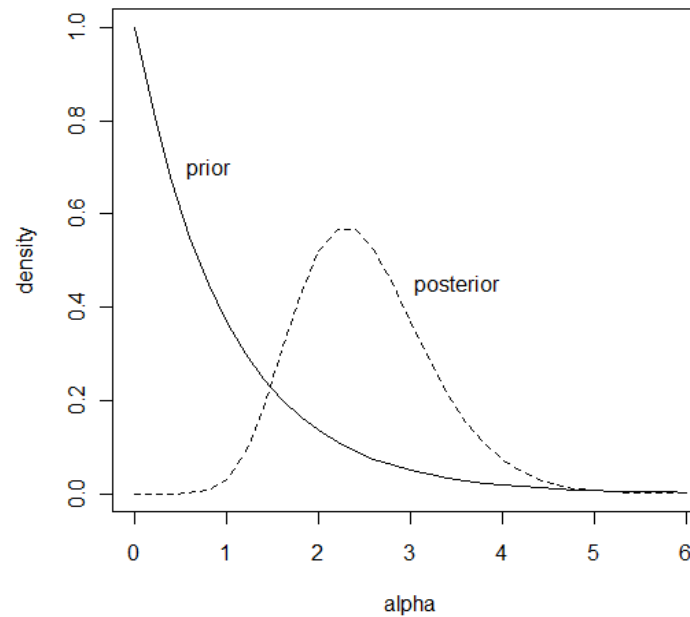
Predictive:

$$f_{Y|\mathbf{X}}(y\,|\,\mathbf{x}) = \int_0^\infty f_{Y|A}(y\,|\,\alpha)\,\pi_{A|\mathbf{X}}(\alpha\,|\,\mathbf{x})\,d\alpha = \int_0^\infty \frac{\alpha\,100^\alpha}{y^{\alpha+1}}\,\frac{\alpha^{11}\,e^{-\alpha/0.20828}}{\Gamma(12)\,0.20828^{12}}\,d\alpha$$

$$= \frac{1}{\Gamma(12)\,0.20828^{12}\,y}\int_0^\infty \alpha^{12} e^{-\alpha\,4.801121 + \alpha\ln 100 - \alpha\ln y}\,d\alpha$$

$$= \frac{1}{\Gamma(12)\,0.20828^{12}\,y}\int_0^\infty \alpha^{12} e^{-\alpha(0.195951 + \ln y)}\,d\alpha \qquad y > 100$$

The integrand is the *core* of a gamma density function with parameters 13 and $1/(0.195951 + \ln y)$.
Then we can use the usual normalizing constant to calculate the integral. We get

$$f_{Y|\mathbf{X}}(y\,|\,\mathbf{x}) = \frac{1}{\Gamma(12)\,0.20828^{12}\,y}\,\frac{\Gamma(13)}{(0.195951 + \ln y)^{13}}\int_0^\infty \frac{(0.195951 + \ln y)^{13}}{\Gamma(13)}\,\alpha^{12} e^{-\alpha(0.195951 + \ln y)}\,d\alpha$$

$$= \frac{12\times(1/0.20828)^{12}}{(0.195951 + \ln y)^{13}\times y} = \frac{4.801121^{12}\times 12}{(0.195951 + \ln y)^{13}\times y} \qquad y > 100$$

The density does not look familiar but it can be proved that $\ln Y - \ln 100$ has a Pareto distribution.

```
> y=seq(100,400,by=20)
> yy1=2*(y^(-1))*((1+log(y/100))^(-3));
> plot(y,yy1,type="l",ylab="predictive density",xlab="y")
> yy2=3.757995*factorial(12)*(y^(-1))*((0.195951+log(y))^(-13));
> lines(y,yy2,type="l",lty=2)
> text(130,0.005,"before"); text(150,0.010,"after")
```
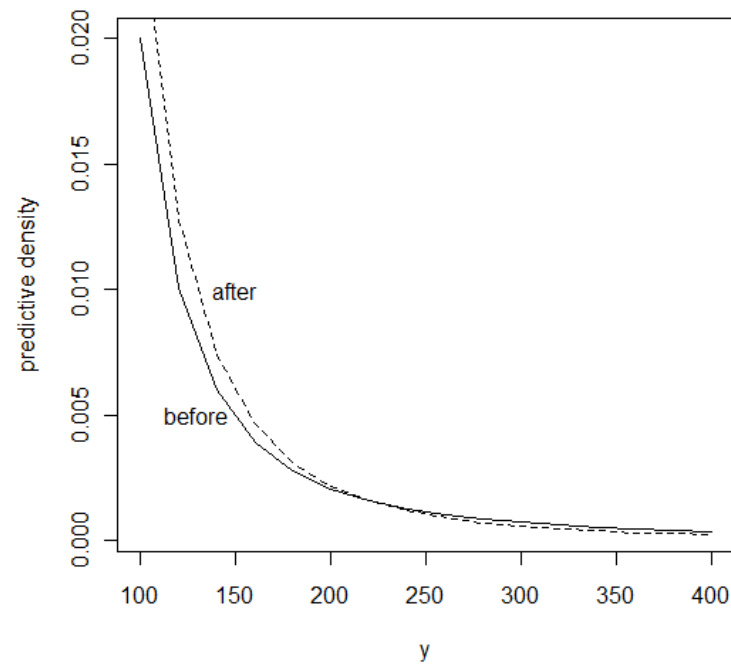
Bayesian inference and prediction

From a Bayesian point of view the analysis is complete when we specify the posterior distribution which quantifies our knowledge about θ after the observation of the sample. However, for practical purposes point estimation and/or "confidence interval" are, most of the time, needed. The problem is how to sum up a distribution in one point or using an interval. For point estimation the usual Bayesian solution is to use a loss function.

○ **Definition 15.9** – A **loss function** $l_j(\hat{\theta}_j, \theta_j)$ describes the penalty paid by the investigator when $\hat{\theta}_j$ is the estimate and θ_j is the true value of the jth parameter.

○ **Comment**: The loss function is random since it depends on θ_j.

○ **Definition 15.10** – The **Bayes estimate** for a given loss function is the one that minimizes the expected loss, given the posterior distribution of the parameter in question.

○ **Definition 15.11** – For **squared-error** loss, the loss function is (all subscripts are dropped for convenience) $l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$. For **absolute loss** it is $l(\hat{\theta}, \theta) = |\hat{\theta} - \theta|$. For **zero-one loss** it is $l(\hat{\theta}, \theta) = 0$ if $\hat{\theta} = \theta$ and 1 otherwise.

○ **Comment**: Strictly speaking, Definition 13.17 defines the loss functions up to a multiplicative constant.

o **Theorem 15.12** – For squared-loss, the Bayes estimate is the mean of the posterior distribution; for absolute loss it is the median and for zero-one loss it is the mode.

Challenging question: Prove the theorem for the squared loss function (easier) and other functions. □

o **Comments**:

    o There is no guarantee that the posterior's mean exists (or the mode) or that the median is unique.

    o When no otherwise specified, the term Bayes estimate refers to the posterior mean (squared-loss function).

o **Example 15.3** – Determine the three estimates of α (example 15.1 continued)

The posterior is a gamma distribution with parameters 12 and 0.20828. Then $E(\alpha\,|\,\mathbf{x}) = 2.49942$, the mode is $11 \times 0.20828 = 2.291132$ and the median has to be determined numerically (2.430342).

o Sometimes the expected value of the predictive distribution is of interest. We can calculate it using the predictive and it can be shown that $E(Y\,|\,\mathbf{x}) = \int y\, f_{Y|\mathbf{X}}(y\,|\,\mathbf{x})\,dy = \int \pi_{\Theta|\mathbf{X}}(\theta\,|\,\mathbf{x})\, E(Y\,|\,\theta)\,d\theta$ (see *Loss Models*).

Bayesian HPD credibility set

o **Definition 15.13** – The points $a < b$ define a $100 \times (1-\alpha)\%$ **credibility interval** for θ_j, provided that

$\Pr(a \leq \theta_j \leq b) \geq 1 - \alpha$.

o **Comments**:

   o The term credibility is used to underline the differences between the frequentist (confidence interval) and the Bayesian approaches. This term has no relation with credibility theory.

   o The inequality is due to discrete distribution

   o Definition 15.19 does not produce a unique solution for the credibility interval. Usually we look for the shortest interval.

o **Theorem 15.14** – If the posterior random variable $\theta_j \mid \mathbf{x}$ is continuous and unimodal, then the $100 \times (1-\alpha)$ credibility interval with the smallest width, $b - a$, is the unique solution to

$$\int_a^b \pi_{\Theta_j \mid \mathbf{X}}(\theta_j \mid \mathbf{x}) \, d\theta = 1 - \alpha \text{ and } \pi_{\Theta_j \mid \mathbf{X}}(b \mid \mathbf{x}) = \pi_{\Theta_j \mid \mathbf{X}}(a \mid \mathbf{x})$$

This interval is a special case of a highest posterior density (HPD).

o **Comment**: The posterior cannot have any local maximum except the mode which is unique.

o **Example 15.5** – Determine the shortest 95% credibility interval for the parameter α (example 15.1 continued)
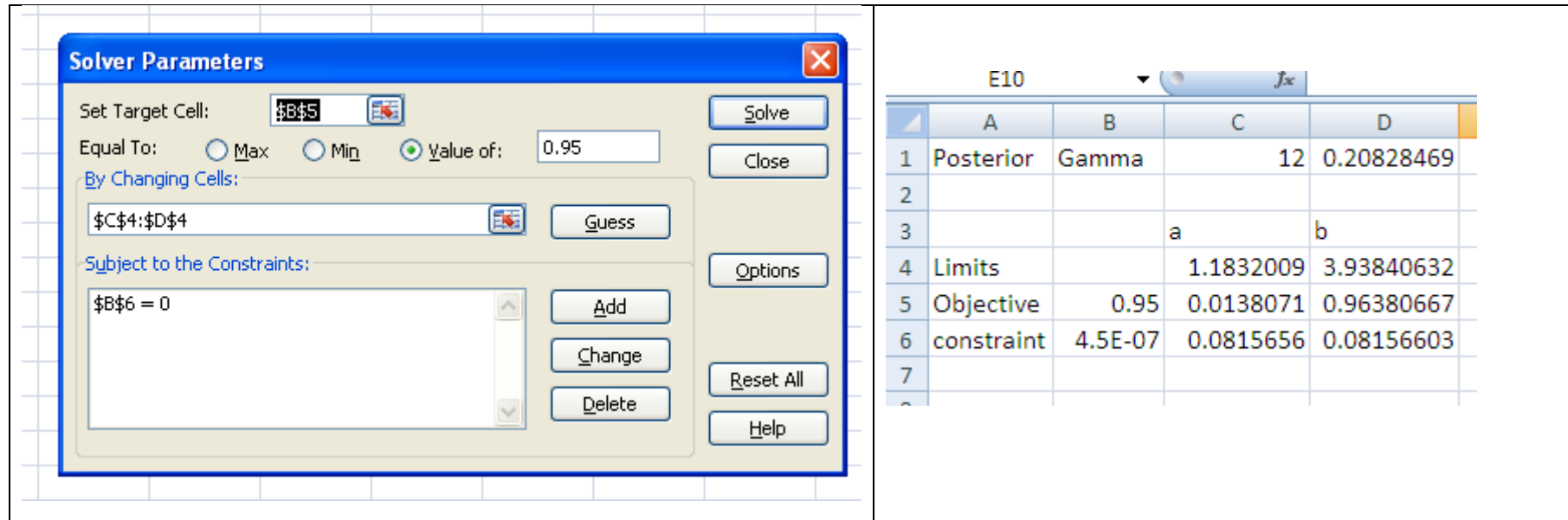
Let us use EXCEL's solver to determine the interval

	A	B	C	D
1	Posterior	Gamma	12	0.20828469
2				
3			a	b
4	Limits		0.5	5
5	Objective	0.99748	8.473E-06	0.99748392
6	constraint	0.00672	0.0001664	0.00688849
7				
8				

In cells C4 and D4 we put two initial values for the limits of the interval

In cells C5 and D5 we compute the value of the distribution function at points *a* and *b* respectively (GAMMA.DIST(C4;C1;D1;1) for C5). Cell B5 contains the probability of the interval.

In cells C6 and D6 we calculate the value of the posterior density function at points *a* and *b* respectively (GAMMA.DIST(C4;C1;D1;**0**) for C6). Cell B6 contains the difference between the density at point *b* and the density at point *a*.

Now we want to get the value $1-\alpha = 0.95$ in cell B5 and the value 0 in cell B6.

	A	B	C	D
1	Posterior	Gamma	12	0.20828469
2				
3			a	b
4	Limits		1.1832009	3.93840632
5	Objective	0.95	0.0138071	0.96380667
6	constraint	4.5E-07	0.0815656	0.08156603
7				

The credibility interval is then (1.1832009; 3.93840632). If needed we can add more constraints.

To get an approximate solution we can place a probability of 0.025 at each tail → (1.29148; 4.09947).

- **Definition 15.15** – For any posterior distribution the $100 \times (1-\alpha)\%$ HPD credibility set is the set of parameter values C such that $\Pr(\theta_j \in C) \geq 1 - \alpha$ and $C = \left\{ \theta_j : \pi_{\Theta_j|\mathbf{X}}(\theta_j \mid \mathbf{x}) \geq c \right\}$ for some *c*, where *c* is the largest value for which the previous inequality holds.

- Comment: The credibility set may be the union of several intervals if the posterior is a multimodal distribution. If the distribution is unimodal we get the HPD interval.

- Sometimes computing posterior probabilities is difficult but computing posterior moments is easier. We can them using the Bayesian central limit theorem.

- **Theorem 15.16 – Bayesian central limit theorem** – If $\pi(\theta)$ and $f_{\mathbf{X}|\Theta}(x \mid \theta)$ are both twice differentiable in the elements of θ and other commonly satisfied assumptions hold, then the posterior distribution of Θ given $\mathbf{X} = \mathbf{x}$ is asymptotically normal.

- **Comment**: The "commonly satisfied assumptions" are like those presented with Theorem 15.5

o **Example 15.6** – Construct a 95% credibility interval for α using the Bayesian central limit theorem (example 15.1 continued).

The posterior is $\gamma(12, 0.20828)$ and then $E(\alpha \mid \mathbf{x}) = 2.49942$ and $\mathrm{var}(\alpha \mid \mathbf{x}) = 0.52059$. The credibility interval is then $2.49942 \pm 1.96 \times \sqrt{0.52059}$, i.e. (1.085238, 3.913594). Note that the method is not appropriate for this example as the sample size is far from large.

Alternatively, we can replace the mean of the posterior by the mode and/or we can use minus the inverse of the 2nd derivative of the log of the posterior to compute the variance.

In our example: Mode → 2.291132

Log of the posterior: $\ln \pi_{\alpha \mid \mathbf{X}}(\alpha \mid \mathbf{x}) = -\ln \Gamma(12) - 12 \ln 0.20828 + 11 \ln \alpha - \alpha / 0.20828$

1st derivative
$$\frac{d}{d\alpha} \ln \pi_{\alpha \mid \mathbf{X}}(\alpha \mid \mathbf{x}) = \frac{11}{\alpha} - \frac{1}{0.20828}$$

2nd derivative
$$\frac{d^2}{d\alpha^2} \ln \pi_{\alpha \mid \mathbf{X}}(\alpha \mid \mathbf{x}) = -\frac{11}{\alpha^2}$$

Variance to be used → $\dfrac{\alpha^2}{11}$